Structure of quasiparticles and their fusion algebra in fractional quantum Hall states

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It was recently discovered that fractional quantum Hall (FQH) states can be characterized quantitatively by the pattern of zeros that describe how the ground-state wave function goes to zero when electrons are brought close together. Quasiparticles in the FQH states can be described in a similar quantitative way by the pattern of zeros that result when electrons are brought close to the quasiparticles. In this paper, we combine the pattern of zeros approach and the conformal field theory (CFT) approach to calculate the topological properties of quasiparticles. We discuss how the quasiparticles in FQH states naturally form representations of a magnetic translation algebra, with members of a representation differing from each other by Abelian quasiparticles. We find that this structure dramatically simplifies topological properties of the quasiparticles, such as their fusion rules, charges, and scaling dimensions, and has consequences for the ground state degeneracy of FQH states on higher genus surfaces. We find constraints on the pattern of zeros of quasiparticles that can fuse together, which allow us to derive the fusion rules of quasiparticles from their pattern of zeros, at least in the case of the (generalized and composite) parafermion states. We also calculate from CFT the number of quasiparticle types in the generalized and composite parafermion states, which confirm the result obtained previously through a completely different approach.

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I. INTRODUCTION

For a long time, people believed that all phases and all continuous phases transitions are described by symmetry breaking and the associated order parameters.[1] However, after the discovery of fractional quantum Hall (FQH) states,[2,3] we realized that the FQH states are described by topological order—another kind of order beyond symmetry breaking.[4] Different FQH states are states with different topological orders, which lead to different statistics of the quasiparticles,[5] different edge states,[6] different topological entanglement entropies,[7,8] etc. Thus the discovery of FQH states really enriches our understanding of orders that quantum many-body systems can have.

Since the topological orders in FQH states cannot be described by order parameters associated with symmetry breaking, it is not clear what is the correct and proper way to describe topological order. As a result, we still refer to FQH states using their filling fractions. This is just like referring to different crystals using their densities, which reflects our poor understanding of topological order. Thus finding new ways to describe and characterize topological order becomes a key issue in developing a theory of topological order.

Recently, it was realized that topological order corresponds to patterns of long-range entanglement.[7–9] Many theoretical studies reveal that the patterns of entanglement in many-body states are extremely rich;[10–13] but, how do we find a systematic and quantitative way to describe all the possible patterns of entanglement? In an attempt to obtain a systematic and quantitative description of topological orders[4] in FQH states,[2,3] it has been shown recently that FQH wave functions can be classified according to their pattern of zeros, as characterized by a sequence of integers $\{S_a\}$, which describes the manner in which ground state wave functions go to zero as the coordinates of various clusters of electrons are brought together.[14] Each quasiparticle can also be character-

ized in a similar quantitative fashion by the pattern of zeros of the wave function as electrons are brought close to quasiparticles. The quasiparticle pattern of zeros is again described by a sequence of integers $\{S_{\gamma,a}\}$ and different quasiparticles are described by different sequences. $\{S_a\}$ and $\{S_{\gamma,a}\}$ represent a quantitative description of topological order in FQH states. Such a quantitative description has been used to calculate some of the topological properties of quantum Hall states, such as the numbers of quasiparticle types and charges of quasiparticles, in a quantitative way.[15]

However, the results in Refs. [14] and [15] are based on certain untested assumptions, so those results need to be confirmed through other independent methods. Some of the FQH wave functions classified in Ref. [14] are equal to correlation functions of certain known conformal field theories (CFTs). For those special FQH states, we can calculate through CFT their topological properties such as the number of types of quasiparticles, their respective electric charges, fusion rules, and spins.[16,17] This allows us to check the results in Refs. [14] and [15], at least for those FQH states that are associated with known CFTs.

In this paper, we will carry out such a calculation and compare the results from CFT with those from the pattern of zeros. We calculate from CFT the total number of the quasiparticle types and their charges for the so-called generalized and composite parafermion states [see Eq. (72)] (Refs. [14], [15], and [18]) and find agreement with results obtained from the pattern of zeros approach.

We also combine the pattern of zeros approach and the CFT approach to study topological properties of FQH states and have obtained results that generalize those in Ref. [15]. We find in general that the pattern of zeros approach gives rise to a natural notion of "translation" that acts on quasiparticles. This allows us to show that quasiparticles in FQH states form a representation of a magnetic translation algebra [see Eq. (27)], with members of each representation differing from each other by Abelian quasiparticles. This is consistent

with the fact that the quasiparticles have a one-to-one correspondence with degenerate FQH ground states on the torus, which form a representation of the magnetic translation algebra. It also implies that various topological properties such as fusion rules and scaling dimensions may simplify dramatically [see Eqs. (35), (40), and (25)]. A special consequence of this structure is that it allows us to prove quite generally that the ground state degeneracy of FQH states on genus g surfaces is given by ν^{-g} times a factor that depends only on the "non-Abelian" part of the CFT and not on the filling fraction ν [see Eq. (54)].

We further discuss fusion rules and their connection to domain walls in the pattern of zeros sequences. There, we find a nontrivial condition on the pattern of zeros of a set of quasiparticles that can be involved in fusion with each other [see Eq. (50)]. In the general and composite parafermion states, this condition is sufficient to completely determine the fusion rules and may perhaps also be sufficient to do so more generally in other FQH states. If the latter is true then one can derive the fusion rules from the pattern of zeros. The pattern of zeros approach allows us to understand the structure of CFT in a more physical way.

In summary, in this paper, we use the pattern of zeros $\{S_a\}$—a quantitative description of the topological order—to calculate various topological properties of FQH states. This is achieved by introducing the quasiparticle pattern of zeros $\{S_{\gamma;a}\}$—a quantitative description of the quasiparticles. We show how to use $\{S_a\}$ and $\{S_{\gamma;a}\}$ to calculate quasiparticle charges, quasiparticle quantum dimensions, quasiparticle fusion algebra, and ground state degeneracies on genus g Riemann surfaces, etc.

In Sec. II, we discuss the close relation between the pattern of zeros approach and the CFT approach. In Sec. III, we combine the pattern of zeros approach and the CFT approach to study the structure of quasiparticles in FQH states. In Sec. IV, we study the parafermion FQH states in detail, which allows us to compare and check the results obtained from the pattern of zeros approach and the CFT approach. In Sec. V, we apply our results to study some concrete examples.

II. PATTERN OF ZEROS AND CONFORMAL FIELD THEORY

A. FQH wave function as a correlation function in CFT

The ground state wave function of a FQH state (in the first Landau level) has a form

$$\Psi = \Phi(\{z_i\})\exp\left(-\tfrac{1}{4}\sum |z_i|^2\right),$$

where $z_i = x_i + iy_i$ is the coordinate of the ith electron. Here $\Phi(\{z_i\})$ is an antisymmetric polynomial (for fermionic electrons) or a symmetric polynomial (for bosonic electrons). In this paper, we will only consider the cases of bosonic electrons where $\Phi(\{z_i\})$ is a symmetric polynomial. The case of fermionic electrons can be included by replacing $\Phi(\{z_i\})$ by $\Phi(\{z_i\})\Pi_{i<j}(z_i - z_j)$.

In Refs. 14 and 15, the symmetric polynomials $\Phi(\{z_i\})$ are studied and classified directly through their pattern of zeros. In this paper, we will study symmetric polynomials through

CFT. This is possible since for a class of ideal FQH states, the symmetric polynomial Φ can be written as a correlation function of vertex operators $V_e(z)$ in a CFT,[16,17,19]

$$\Phi(\{z_i\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_N}\left\langle V(z_\infty)\prod_i V_e(z_i)\right\rangle. \tag{1}$$

Such a relation allows us to study and classify FQH states through a study and a classification of proper CFTs.

In the above expression, V_e (which will be called an electron operator) has a form

$$V_e(z) = \psi(z)e^{i\varphi(z)/\sqrt{\nu}},$$

where ν is the filling fraction of the FQH state. The CFT generated by the V_e operator contains two parts. The first part—the simple-current part—is generated by a simple-current operator ψ, which satisfies an Abelian fusion rule[20,21]

$$\psi_a(z)\psi_b(z) = \psi_{a+b}(z), \quad \psi_a(z) \equiv [\psi(z)]^a.$$

The second part—the U(1) "charge" part—is generated by the vertex operator $e^{i\varphi(z)/\sqrt{\nu}}$ of a Gaussian model, which has a scaling dimension $h = \frac{1}{2\nu}$. The scaling dimension of ψ_a is denoted as $h_a^{\rm sc}$. Thus the scaling dimension of the ath power of the electron operator

$$V_a \equiv (V_e)^a = \psi_a e^{ia\varphi(z)/\sqrt{\nu}}$$

is given by

$$h_a = h_a^{\rm sc} + \frac{a^2}{2\nu}. \tag{2}$$

B. Pattern of zeros approach and CFT approach

In Ref. 14, a pattern of zeros $\{S_a\}$ is introduced to characterize a FQH state, where the integer S_a is defined as

$$\Phi(\{z_i\})|_{\lambda \to 0} = \lambda^{S_a}P(\xi_1, \cdots, \xi_a; z_{a+1}, \cdots) + \cdots,$$

where $z_i = \lambda\xi_i$, and $i = 1, \ldots, a$. In other words, S_a is the order of zeros in $\Phi(\{z_i\})$ as we bring a electrons together. The pattern of zeros characterization also applies to FQH states generated by CFT, so in this section we will discuss the relation between the CFT approach and the pattern of zeros approach in a general setting.

In the pattern of zeros approach, a FQH state is characterized by the sequence $\{S_a\}$. In the CFT approach, a FQH state is characterized by the sequence $\{h_a\}$ or equivalently $\{h_a^{\rm sc}\}$. From the operator product expansion (OPE) of the electron operators

$$V_a(z)V_b(w) = \frac{C_{abc}}{(z-w)^{h_a+h_b-h_c}}V_c(w) + \cdots, \tag{3}$$

we find that $\{S_a\}$ and $\{h_a\}$ are closely related

$$S_a = h_a - ah_1. \tag{4}$$

In Ref. 14, it was shown that $\{S_a\}$ should satisfy

$$\Delta_2(a,a) = \text{even} \geq 0, \quad \Delta_3(a,b,c) = \text{even} \geq 0, \tag{5}$$

where

$$\Delta_2 \equiv S_{a+b} - S_a - S_b,$$

$$\Delta_3 \equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c. \quad (6)$$

Finding the sequences $\{S_a\}$ that satisfy the above conditions allows us to obtain a classification of symmetric polynomials and FQH states.

The condition (5) becomes the following conditions on h_a^{sc}:

$$\Delta_2^{\mathrm{sc}}(a,b) + \frac{ab}{\nu} = \text{integer} \geq 0,$$

$$\Delta_2^{\mathrm{sc}}(a,a) + \frac{a^2}{\nu} = \text{even}, \quad \Delta_3^{\mathrm{sc}}(a,b,c) = \text{even} \geq 0, \quad (7)$$

where

$$\Delta_2^{\mathrm{sc}} = h_{a+b}^{\mathrm{sc}} - h_a^{\mathrm{sc}} - h_b^{\mathrm{sc}},$$

$$\Delta_3^{\mathrm{sc}} = h_{a+b+c}^{\mathrm{sc}} - h_{a+b}^{\mathrm{sc}} - h_{b+c}^{\mathrm{sc}} - h_{a+c}^{\mathrm{sc}} + h_a^{\mathrm{sc}} + h_b^{\mathrm{sc}} + h_c^{\mathrm{sc}}.$$

It is not surprising to see that the equations in Eq. (7) are actually a part of the defining conditions of parafermion CFTs.[20,21] This reveals a close connection between the CFT approach and the pattern of zeros approach of FQH states. This also explains why many FQH states obtained from the pattern of zeros construction are related to parafermion FQH states.

After understanding the relation between the pattern of zeros approach and the CFT approach, we are able to consider in more detail an important issue of stability. In the pattern of zeros approach, we use a sequence of integers $\{S_a\}$ to characterize a FQH state. The question is, does the sequence $\{S_a\}$ uniquely determine the FQH state? Can there be more than one FQH states that give rise to the same pattern of zeros? Through a few examples, we find that some sequences $\{S_a\}$ uniquely determine the corresponding FQH states, while other sequences $\{S_a\}$ cannot determine the FQH state uniquely. Through the relation to the CFT, we can address such a question from another angle. We would like to ask, can the scaling dimensions h_a of the simple currents V_a uniquely determine the correlation function of those operators? Or more simply, can the scaling dimensions h_a of the simple currents V_a uniquely determine the structure constants C_{abc} in the OPE of the simple-current operators [see Eq. (3)]? Such a question has been studied partially in CFT. It was shown[20] that if $h_a^{\mathrm{sc}} = a(n-a)/n$, then C_{abc} is uniquely determined. On the other hand, if $h_a^{\mathrm{sc}} = 2a(n-a)/n$ then C_{abc} can depend on a continuous parameter. In this case, the pattern of zeros cannot uniquely determine the FQH wave function. We may have many linearly independent wave functions (even on a sphere) that have the same pattern of zeros.

C. Pattern of zeros of the quasiparticle operators in CFT

The state Φ_γ with a quasiparticle at ξ can also be expressed as a correlation function in a CFT,

$$\Phi_\gamma(\xi;\{z_i\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_N^q} \left\langle V_q(z_\infty) V_\gamma(\xi) \prod_i V_e(z_i) \right\rangle. \quad (8)$$

Here V_γ is a quasiparticle operator in the CFT and has a form

$$V_\gamma(z) = \sigma_\gamma(z) e^{i\varphi(z)Q_\gamma/\sqrt{\nu}}, \quad (9)$$

where $\sigma_\gamma(z)$ is a "disorder" operator in the CFT generated by the simple-current operator ψ. Different quasiparticles labeled by different γ will correspond to different "disorder" operators. Q_γ is the charge of the quasiparticle.

How can we obtain the properties, such as the charge Q_γ, of the quasiparticles? It is hard to proceed from the abstract symbol γ which actually contains no information about the quasiparticle. It turns out that the pattern of zeros provides a quantitative way to characterize the quasiparticle operator. Such a quantitative characterization does contain information about the quasiparticle and will help us calculate its properties.

To obtain the quantitative characterization, we first fuse the quasiparticle operator with a electron operators,

$$V_{\gamma+a}(z) = V_\gamma V_a = \sigma_{\gamma+a}(z) e^{i\varphi(z)Q_{\gamma+a}/\sqrt{\nu}},$$

$$\sigma_{\gamma+a} = \sigma_\gamma \psi_a, \quad Q_{\gamma+a} = Q_\gamma + a. \quad (10)$$

Then, we consider the OPE of $V_{\gamma+a}$ with V_e

$$V_e(z) V_{\gamma+a}(w) = (z-w)^{l_{\gamma,a+1}} V_{\gamma+a+1}(w). \quad (11)$$

Let h_a, h_γ, and $h_{\gamma+a}$ be the scaling dimensions of V_a, V_γ, and $V_{\gamma+a}$, respectively. We have

$$l_{\gamma,a+1} = h_{\gamma+a+1} - h_{\gamma+a} - h_1. \quad (12)$$

Since the quasiparticle wave function $\Phi_\gamma(\{z_i\})$ must be a single-valued function of the z_i's, $l_{\gamma,a}$ must be integer. For the wave function to be finite, $l_{\gamma,a}$ must be non-negative. The sequence of integers $\{l_{\gamma,a}\}$ gives us a quantitative way to characterize quasiparticle operators V_γ in CFT. $\{l_{\gamma,a}\}$ turns out to be exactly the sequence of integers introduced in Ref. 15 to characterize quasiparticles in a FQH state. The sequence $\{l_{\gamma,a}\}$ describes the pattern of zeros for the quasiparticle γ.

According to Ref. 15, not all sequences $\{l_{\gamma,a}\}$ describe valid quasiparticles. The sequences $\{l_{\gamma,a}\}$ that describe valid quasiparticles must satisfy

$$S_{\gamma,a+b} - S_{\gamma,a} - S_b \geq 0,$$

$$S_{\gamma,a+b+c} - S_{\gamma,a+b} - S_{\gamma,a+c} - S_{b+c} + S_{\gamma,a} + S_b + S_c \geq 0, \quad (13)$$

where the integers $S_{\gamma,a}$ are given by

$$S_{\gamma,a} = \sum_{i=1}^{a} l_{\gamma,i} = h_{\gamma+a} - h_\gamma - ah_1. $$

The solutions of Eq. (13) give us the sequences that correspond to all the quasiparticles.

There is an equivalent way to describe the pattern of zeros $\{l_{\gamma,a}\}$ using an occupation-number sequence. Consider a one-dimensional lattice whose sites are labeled by a non-negative

integer l. We can think of $l_{\gamma;a}$ as defining the location of the ath electron on the one-dimensional lattice. Thus the sequence $\{l_{\gamma,a}\}$ describes a pattern of occupation of electrons in the one-dimensional lattice. Such a pattern of occupation can also be described by occupation numbers $\{n_{\gamma,l}\}$, where $n_{\gamma,l}$ denotes the number of electrons at site l. Thus, each quasiparticle V_γ defines a sequence $\{l_{\gamma,a}\}$ and an occupation-number sequence $\{n_{\gamma,l}\}$. The occupation-number sequence $\{n_{\gamma,l}\}$ happens to be the same sequence that characterizes the ground states in the thin cylinder limit for the FQH states.[22–24]

The distinct quasiparticles are actually equivalence classes of fields, where two fields are said to belong to the same quasiparticle class (or type) if they differ by an electron operator $V_\gamma \sim V_\gamma V_e$. There are a finite number of these quasiparticle classes, and this number is an important characterization of a topological phase. Two equivalent quasiparticles which are related by a number of electron operators will have nearly the same occupation-number sequence. The quasiparticle operator $V_{\gamma+b} = V_\gamma V_b$ is described by

$$l_{\gamma+b;a} = h_{\gamma+b+a} - h_{\gamma+a+b-1} - h_1 = l_{\gamma,a+b}. \tag{14}$$

Thus if two sequences $\{l_{\gamma,a}\}$ and $\{l_{\gamma';a}\}$ satisfy $l_{\gamma';a} = l_{\gamma,a+b}$, then $V_{\gamma'} = V_\gamma V_b$ and therefore they belong to the same quasiparticle class because they only differ by electron operators. Two such sequences will give occupation-number sequences $\{n_{\gamma,l}\}$ that are the same asymptotically as l grows large but are different near the beginning of the sequence. Thus we can classify the quasiparticle types by the asymptotic form of their occupation-number sequence.

Here we take the point of view that two operators are physically distinct only if their disparity can be resolved by the electron operator. In other words, if two operators in the conformal field theory yield the same pattern of zeros as defined above then the electron operator cannot distinguish between them and therefore we identify them as the same physical operator. This point of view is correct if the pattern of zeros uniquely determines the correlation functions (such as the structure constants C_{abc}).

Let us use $\gamma = 0$ to label the "trivial" quasiparticle created by $V_0 = 1$. We see that such a trivial quasiparticle is characterized by

$$l_{0;a+1} \equiv l_{a+1} = h_{a+1} - h_a - h_1. \tag{15}$$

Since $h_0 = 0$, we see that $l_1 = 0$.

For the FQH states of n-cluster form,[14,15] the corresponding CFT satisfies

$$\psi_n = (\psi)^n = 1. \tag{16}$$

As a result of this cyclic Z_n structure, the scaling dimensions of the simple currents satisfy

$$h_{kn}^{\rm sc} = 0, \quad h_{a+n}^{\rm sc} = h_a^{\rm sc}, \tag{17}$$

where k is a positive integer. Let

$$m \equiv l_{n+1} = h_{n+1} - h_1 - h_n.$$

Using $h_{n+1} - h_1 - h_n = \frac{(n+1)^2 - n^2 - 1}{2\nu} = \frac{n}{\nu}$, we find that the filling fraction ν is given by

$$\nu = \frac{n}{m}. \tag{18}$$

For such a filling fraction, we also find that $l_{\gamma;a}$ satisfies

$$l_{\gamma,a+n} = l_{\gamma,a} + m. \tag{19}$$

This is an important consequence of the Z_n structure. It implies that the occupation numbers $n_{\gamma,l}$ are periodic: $n_{\gamma,l+m} = n_{\gamma,l}$, with a fixed number of particles per unit cell. From the preceding equation, it follows that the size of the unit cell is m and there are n particles in each unit cell.

We also note that, according to numerical experiment,[14] for $h_a^{\rm sc}$ that satisfies Eq. (7), m and S_a must be even, and the solutions satisfy

$$nh_a^{\rm sc} = {\rm integer}.$$

D. Quasiparticle charge from its pattern of zeros

Now let us calculate the quasiparticle charge Q_γ [see Eq. (9)] from the sequence $\{l_{\gamma,a}\}$. Since $\sigma_{\gamma+n} = \sigma_\gamma$, we have [see Eqs. (2) and (12)]

$$h_{\gamma+n} - h_\gamma = \frac{(Q_\gamma + n)^2 - Q_\gamma^2}{2\nu} = nh_1 + \sum_{a=1}^{n} l_{\gamma,a}. \tag{20}$$

Using $Q_{\gamma=0} = 0$, we find a formula for the charge of the quasiparticle in terms of the pattern of zeros,

$$Q_\gamma = \frac{1}{m} \sum_{a=1}^{n} (l_{\gamma,a} - l_a), \tag{21}$$

which agrees with the result obtained in Ref. 15.

III. STRUCTURE OF QUASIPARTICLES

A. Labeling scheme

Let $h_{\gamma+a}^{\rm sc}$ be the scaling dimension of $\sigma_\gamma \psi_a$, which satisfies

$$h_{\gamma+n}^{\rm sc} = h_\gamma^{\rm sc}.$$

Following Eq. (12), we can define a new sequence $\{l_{\gamma,a}^{\rm sc}\}$ that does not depend on the U(1) sector of the CFT and describes the simple-current part of the quasiparticle,

$$l_{\gamma,a+1}^{\rm sc} = h_{\gamma+a+1}^{\rm sc} - h_{\gamma+a}^{\rm sc} - h_1^{\rm sc}. \tag{22}$$

$l_{\gamma,a}^{\rm sc}$ has the following nice properties:

$$l_{\gamma,a+n}^{\rm sc} = l_{\gamma,a}^{\rm sc}, \quad l_{\gamma+b;a}^{\rm sc} = l_{\gamma,a+b}^{\rm sc}.$$

Since $h_{\gamma+a}^{\rm sc} = h_{\gamma+a} - \frac{(Q_\gamma+a)^2}{2\nu}$, $l_{\gamma,a}^{\rm sc}$ and $l_{\gamma,a}$ are related,

$$l_{\gamma,a}^{\rm sc} = l_{\gamma,a} - \frac{m(Q_\gamma + a - 1)}{n}. \tag{23}$$

We see that

$$nl_{\gamma,a}^{\rm sc} = {\rm integer}.$$

We also see that

$$h_{\gamma+a}^{sc} = h_{\gamma}^{sc} + a h_1^{sc} + \sum_{b=1}^{a} l_{\gamma;b}^{sc}.$$

In particular, setting $a=n$ in the preceding equation implies that the average over a complete period of $l_{\gamma;a}^{sc}$ yields the scaling dimension of the simple-current operator,

$$\frac{1}{n}\sum_{b=1}^{n} l_{\gamma;b}^{sc} = -h_1^{sc}.$$

It is convenient to subtract off this average to introduce $\tilde{l}_{\gamma;a}^{sc}$,

$$\tilde{l}_{\gamma;a}^{sc} \equiv l_{\gamma;a}^{sc} + h_1^{sc} = h_{\gamma+a+1}^{sc} - h_{\gamma+a}^{sc},$$

which also satisfies

$$n\tilde{l}_{\gamma;a}^{sc} = \text{integer}. \tag{24}$$

We find that $\tilde{l}_{\gamma;a}^{sc}$ satisfies $\Sigma_{a=1}^{n} \tilde{l}_{\gamma;a}^{sc} = 0$ [see Eq. (22)] and

$$h_{\gamma+a}^{sc} = h_{\gamma}^{sc} + \sum_{b=1}^{a} \tilde{l}_{\gamma;b}^{sc}. \tag{25}$$

We see that if $\sigma_{\gamma'}$ and σ_{γ} are related by a simple-current operator $\sigma_{\gamma'} = \sigma_{\gamma+a} = \sigma_{\gamma}\psi_a$ then the scaling dimension of $\sigma_{\gamma'}$ can be calculated from that of σ_{γ} using Eq. (25).

We have seen that the different quasiparticles for an n-cluster FQH state are labeled by $l_{\gamma,a}$, $a=1,\ldots,n$. In the following, we will show that we can also use $\{Q_{\gamma}; \tilde{l}_{\gamma;1}^{sc}, \cdots, \tilde{l}_{\gamma;n}^{sc}\}$ to label the quasiparticles.

Since $h_{\gamma=0}^{sc} = 0$, from Eq. (25), we see that

$$h_1^{sc} = \tilde{l}_{0;1}^{sc} \equiv \tilde{l}_1^{sc}.$$

Therefore, from Eq. (23), we see that

$$l_{\gamma;a} = \tilde{l}_{\gamma;a}^{sc} - h_1^{sc} + \frac{m(Q_{\gamma} + a - 1)}{n} = \tilde{l}_{\gamma;a}^{sc} - \tilde{l}_1^{sc} + \frac{m(Q_{\gamma} + a - 1)}{n}.$$

So, the quasiparticles can indeed be labeled by $\{Q_{\gamma}; \tilde{l}_{\gamma;1}^{sc}, \cdots, \tilde{l}_{\gamma;n}^{sc}\}$.

We note that $\gamma+1$ corresponds to a bound state between a γ quasiparticle and an electron. The $(\gamma+1)$ quasiparticle is labeled by

$$\{Q_{\gamma+1}; \tilde{l}_{\gamma+1;1}^{sc}, \cdots, \tilde{l}_{\gamma+1;n}^{sc}\} = \{Q_{\gamma} + 1; \tilde{l}_{\gamma;2}^{sc}, \cdots, \tilde{l}_{\gamma;n}^{sc}, \tilde{l}_{\gamma;1}^{sc}\}.$$

Since two quasiparticles that differ by an electron are regarded as equivalent, we can use the above equivalence relation to pick an equivalent label that satisfies $0 \le Q_{\gamma} < 1$. For each equivalence class, there exists only one such label. We also see that the two sequences $\{\tilde{l}_{\gamma;1}^{sc}, \cdots, \tilde{l}_{\gamma;n}^{sc}\}$ for two equivalent quasiparticles only differ by a cyclic permutation.

We would like to point out that two quasiparticles with the same sequence $\{\tilde{l}_{\gamma;1}^{sc}, \cdots, \tilde{l}_{\gamma;n}^{sc}\}$ but different Q_{γ} only differ by a U(1) charge part. This is because $\{\tilde{l}_{\gamma;1}^{sc}, \cdots, \tilde{l}_{\gamma;n}^{sc}\}$ do not depend on the U(1) part of the CFT. They only depend on the simple-current part of CFT. Using the terminology of FQH physics, the above two quasiparticles only differ by an Abe-

lian quasiparticle created by inserting a few units of magnetic flux. Inserting a unit of magnetic flux generates a shift in the occupation number $n_{\gamma;l} \rightarrow n_{\gamma';l} = n_{\gamma;l-1}$.

At this stage, and for what follows, it is helpful to see some examples as described in Table I. The $\nu=1/2$ Z_2 parafermion state has six types of quasiparticles. We see that the six quasiparticle types in the $\nu=1/2$ Z_2 parafermions states are labeled by $\{Q_{\gamma}; \tilde{l}_{\gamma;1}^{sc}, \tilde{l}_{\gamma;2}^{sc}\} = \{0; \frac{1}{2}, -\frac{1}{2}\}$, $\{\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\}$, $\{0; -\frac{1}{2}, \frac{1}{2}\}$, $\{\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\}$, $\{\frac{1}{4}; 0, 0\}$, and $\{\frac{3}{4}; 0, 0\}$. $\{0; \frac{1}{2}, -\frac{1}{2}\}$ is the trivial quasiparticle (i.e., the ground state with no excitation). $\{\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\}$ is an Abelian quasiparticle created by inserting a unit flux quantum. $\{0; -\frac{1}{2}, \frac{1}{2}\}$ is a neutral fermionic quasiparticle created by inserting two unit flux quantum and combining with an electron. $\{\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\}$ is the bound state of the neutral fermionic quasiparticle with the quasiparticle created by inserting a unit flux quantum. $\{\frac{1}{4}; 0, 0\}$ is a non-Abelian quasiparticle. $\{\frac{3}{4}; 0, 0\}$ is the bound state of the above non-Abelian quasiparticle with the quasiparticle created by inserting a unit flux quantum.

B. Magnetic translation algebra

We saw that the distinct quasiparticle classes can be classified by the asymptotic form of the occupation-number sequences $\{n_{\gamma,l}\}$. Asymptotically, $\{n_{\gamma,l}\}$ is periodic, $n_{\gamma,l+m} = n_{\gamma,l}$ for large l, so distinct quasiparticle types can actually be classified by the asymptotic form of a single unit cell; $\{n_{\gamma,am}, n_{\gamma,am+1}, \cdots, n_{\gamma,am+m-1}\}$ for large enough a. Henceforth, we will drop the term am in the subscript, with the understanding that

$$|\gamma\rangle \equiv \{n_{\gamma;0}, n_{\gamma;1}, \cdots, n_{\gamma;m-1}\} \tag{26}$$

refers to the asymptotic form of a single unit cell of the occupation-number sequence $\{n_{\gamma,l}\}$.

In terms of the sequence (26), there is a natural unitary operation of translation that can be defined. In fact, we shall see that the distinct quasiparticle types, when represented using Eq. (26), naturally form representations of the magnetic translation algebra. We are familiar with this phenomenon in quantum Hall systems because the Hamiltonian has the symmetry of the magnetic translation group. Remarkably, this structure already exists in the conformal field theory.

Let us define two "translation" operators $\hat{T}_1$ and $\hat{T}_2$ that act on $\{n_{\gamma;0}, n_{\gamma;1}, \cdots, n_{\gamma;m-1}\}$ in the following way:

$$\hat{T}_1|\gamma\rangle = \hat{T}_1|\{n_{\gamma;0}, n_{\gamma;1}, \cdots, n_{\gamma;m-1}\}\rangle$$
$$= |\{n_{\gamma;m-1}, n_{\gamma;0}, \cdots, n_{\gamma;m-2}\}\rangle = |\gamma'\rangle,$$

$$\hat{T}_2|\gamma\rangle = e^{i2\pi Q_{\gamma}}|\gamma\rangle. \tag{27}$$

Note that the label γ refers to a single representative of an entire equivalence class of quasiparticles and that while all members of the same class will be described by the same set of integers in Eq. (26), their electric charges will differ by integer units, making $e^{i2\pi Q_{\gamma}}$ independent of the specific representative γ and dependent only on the equivalence class to which it belongs.

TABLE I. Pattern of zeros sequences and scaling dimensions defined thus far for the two simplest parafermion states: the Pfaffian states at $\nu=1$ (for bosonic electrons) and $\nu=1/2$ (for fermionic electrons) with $n=2$. The asymptotic form of the occupation-number sequences $\{n_{\gamma;l}\}$ in a single unit cell are listed. $\tilde{l}^{sc}_{\gamma,a}$ is shown for $a=1,2$. $\{l,m\}$ are the SU(2) labels for the parafermion primary fields; for further explanation of the notation, see Sec. IV. For the $\nu=1$ $Z_2^{(1)}$ state, $q=1$ (where $\nu=p/q$ with p and q coprime) and the three quasiparticles form a dimension 2 and dimension 1 representation of $(\hat{T}_1,\hat{T}_2)$, as one can see from the action of $\hat{T}_1$, which cyclically permutes $\{n_{\gamma,l}\}$. For the $\nu=1/2$ $Z_2^{(1)}$ state, $q=2$ and the six quasiparticles form a dimension 4 and dimension 2 representation of $(\hat{T}_1,\hat{T}_2)$. (The dimension 4 representation is not an irreducible representation. There are three irreducible representations in both cases.)

$\{Q_\gamma;l,m\}$	$n_{\gamma,l}$	$n\tilde{l}^{sc}_{\gamma;a}$	h_γ	h^{sc}_γ	$h^{sc}_{\gamma,\min}$
		$Z_2^{(1)}$ $\nu=1$			
$\{0;0,0\}$	20	1 −1	0	0	0
$\{0;0,2\}$	02	−1 1	1/2	1/2	0
$\{1/2;1,1\}$	11	0 0	3/16	1/16	1/16
		$Z_2^{(1)}$ $\nu=1/2$			
$Q_\gamma;l,m$	$n_{\gamma,l}$	$n\tilde{l}^{sc}_{\gamma;a}$	h_γ	h^{sc}_γ	$h^{sc}_{\gamma,\min}$
$\{0;0,0\}$	1100	1 −1	0	0	0
$\{1/2;0,0\}$	0110	1 −1	1/4	0	0
$\{0;0,2\}$	0011	−1 1	1/2	1/2	0
$\{1/2;0,2\}$	1001	−1 1	3/4	1/2	0
$\{1/4;1,1\}$	1010	0 0	1/8	1/16	1/16
$\{3/4;1,1\}$	0101	0 0	5/8	1/16	1/16

In terms of the $\{l_{\gamma,a}\}$ sequence, Eq. (27) implies that the sequence for γ' is closely related to that for γ plus some number b of electrons,

$$l_{\gamma';a} = l_{\gamma+b;a} + 1, \tag{28}$$

where b depends on which specific representative γ' is chosen from the equivalence class that contains it. Equation (28) implies, from Eq. (21), that the charges $Q_\gamma=Q_{\gamma+b}-b$ and $Q_{\gamma'}$ are related,

$$Q_{\gamma'} - Q_\gamma = \frac{1}{m}\sum_{a=1}^{n}(l_{\gamma';a} - l_{\gamma+b;a}) + b = \frac{n}{m} + b. \tag{29}$$

This means that modulo 1, γ, and γ' differ in charge by ν. From the above relations, we can deduce that $\hat{T}_1$ and $\hat{T}_2$ satisfy the magnetic translation algebra,

$$\hat{T}_2\hat{T}_1 = \hat{T}_1\hat{T}_2 e^{2\pi i\nu}. \tag{30}$$

The key distinction between quasiparticles in different representations of the above magnetic algebra is that they may differ in their non-Abelian content. They can be made of different disorder operators σ_γ, which are non-Abelian operators in the sense that when σ_γ and $\sigma_{\gamma'}$ are fused together, the result may be a sum of several different operators. In contrast, quasiparticles that belong to the same representation differ from each other by only an Abelian quasiparticle.

This can be seen as follows. For two quasiparticles γ and γ' whose occupation-number sequences are related by a translation $\hat{T}_1$, we have, according to Eq. (28)—$l_{\gamma';a}=l_{\gamma+b;a}+1$. It is easily verified in this case that the simple-current part of their pattern of zeros is the same up to a cyclic permutation $l^{sc}_{\gamma';a}=l^{sc}_{\gamma+b;a}=l^{sc}_{\gamma,a+b}$, which implies that γ and γ' are both made of the same disorder operator σ_γ. It can also be verified that $\nu=(Q_{\gamma'}-Q_{\gamma+b})$. So, modulo electron operators, the difference between γ and γ' is solely a U(1) factor. That is, if $\hat{T}_1|\gamma\rangle=|\gamma'\rangle$ then the pattern of zeros of the operator $\sigma_\gamma e^{i(Q_\gamma+\nu)\sqrt{1/\nu}\varphi}$ is described by $|\gamma'\rangle$. We may later abuse this notation and refer to $\hat{T}_1$ as acting on a quasiparticle operator V_γ to give another quasiparticle $V_{\gamma'}=V_\gamma e^{i\nu\sqrt{1/\nu}\varphi}$, by which we mean that $\hat{T}_1$ acts on the pattern of zeros of V_γ and yields the pattern of zeros of $V_{\gamma'}$.

This structure has important consequences for the topological properties of the quasiparticles. Let the filling fraction have a form $\nu=p/q$ where p and q are coprime. Each quasiparticle must belong to a representation of the magnetic translation algebra generated by $\hat{T}_1$ and $\hat{T}_2$. The dimension of each representation is an integer multiple of q (see Table I). This is because two quasiparticles related by the action of $\hat{T}_1$ differ in charge (modulo 1) by ν, and therefore we come back to the same quasiparticle if and only if we apply $\hat{T}_1$ a multiple of q times. The dimension of each representation is at most m (where recall $m\equiv l_{n+1}$ is the size of the unit cell of

the occupation-number sequences and $\nu = n/m$).

Let us relabel the quasiparticle γ as (i, α), with the Roman index i labeling the representation and the Greek index $\alpha \in \mathbb{Z}_{c_i q}$ labeling the particular quasiparticle within the ith representation. c_i is an integer and $c_i q$ is the dimension of the ith representation. (i, α) and $(i, \alpha + c_i q)$ refer to the same quasiparticle. We can choose the labels α such that

$$\hat{T}_1 |i, \alpha\rangle = |i, \alpha + 1\rangle, \tag{31}$$

and this implies that the quasiparticle operator $V_{i,\alpha}$ is related (modulo electron operators) to $V_{i,\alpha+1}$ by a U(1) factor,

$$V_{i,\alpha+1} = e^{i\nu(1/\sqrt{\nu})\varphi} V_{i,\alpha}. \tag{32}$$

In terms of the charges, this is equivalent to writing

$$Q_{(i,\alpha+1)} \bmod 1 = (Q_{(i,\alpha)} + \nu) \bmod 1. \tag{33}$$

Note that we consider the charge modulo one because of the equivalence of two quasiparticles that are related by electron operators.

In this notation, we can write the fusion rules as

$$V_{i,\alpha} V_{j,\beta} = \sum_{k,\gamma} N^{(k,\gamma)}_{(i,\alpha),(j,\beta)} V_{k,\gamma}. \tag{34}$$

The magnetic algebra structure of the quasiparticles implies an important simplification in the fusion rules,

$$N^{(k,\gamma)}_{(i,\alpha),(j,\beta)} = N^{(k,\gamma-\alpha-\beta)}_{(i,0),(j,0)}. \tag{35}$$

This means that the fusion rules for all of the quasiparticles are determined by the much smaller set of numbers given by $N^{(k,\delta)}_{(i,0),(j,0)}$. Furthermore, since charge is conserved in fusion, $N^{(k,\delta)}_{(i,0),(j,0)} = 0$ if $(Q_{(i,0)} + Q_{(j,0)} - Q_{(k,\delta)}) \bmod 1 \neq 0$. There are only c_k different quasiparticles in the kth representation that have the same charge modulo 1; so for each i, j, and k, there are actually only c_k different values of δ for which $N^{(k,\delta)}_{(i,0),(j,0)}$ must be specified. In particular, knowing that a quasiparticle from k is produced in the fusion of $(i,0)$ and $(j,0)$ is generally not enough information to completely specify the fusion rules. However, in some cases, even more information can be massaged out of these relations.

The ith representation has dimension $c_i q$, from which it follows that $(i, c_i q)$ and $(i, 0)$ label the same quasiparticle. From Eq. (35), we can deduce the following identity:

$$N^{(k,\delta)}_{(i,0),(j,0)} = N^{(k,\delta)}_{(i,c_i q),(j,0)} = N^{(k,\delta-c_i q)}_{(i,0),(j,0)}. \tag{36}$$

Suppose that there are integers n, m, and l for which

$$n c_i + m c_j + l c_k = 1. \tag{37}$$

This happens when the greatest common divisor (gcd) of c_i, c_j, and c_k is 1. In this case, using Eq. (36), one finds

$$N^{(k,\delta)}_{(i,0),(j,0)} = N^{(k,\delta+q)}_{(i,0),(j,0)}. \tag{38}$$

This means that if one quasiparticle from the kth representation is produced from fusion of $(i,0)$ and $(j,0)$ then all quasiparticles with the same charge are also produced. In particular, if $\gcd(c_i, c_j, c_k) = 1$ for all choices of i, j, and k, which happens when $m = q$ then the fusion rules are completely specified by the way different representations of the mag-

netic algebra fuse together. We can conclude that when $m = q$, the representations of the magnetic algebra are all irreducible and the fusion rules decompose in the following way:

$$N^{(k,\gamma)}_{(i,\alpha),(j,\beta)} = \begin{cases} \bar{N}^k_{i,j} & \text{if } (Q_{(i,\alpha)} + Q_{(j,\beta)} - Q_{(k,\gamma)}) \% 1 = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

More generally, it is straightforward to check that

$$N^{(k,\delta)}_{(i,0),(j,0)} = N^{[k,\delta+\gcd(c_i,c_j,c_k)q]}_{(i,0),(j,0)}, \tag{40}$$

which implies that once i, j, and k are fixed, the fusion rules are completely specified by $\gcd(c_i, c_j, c_k)$ of the fusion coefficients. The rest of the fusion coefficients can be obtained from Eqs. (35) and (36). As a special familiar example of this, consider the Pfaffian quantum Hall states at $\nu = 1/q$. There, the quasiparticles form two representations of the magnetic translation algebra: one with dimension q, and the other with dimension $2q$, for a total of $3q$ quasiparticles. The quasiparticles in the dimension $2q$ representation are $e^{il\varphi/\sqrt{q}}$ and $\psi e^{il\varphi/\sqrt{q}}$ for $l = 0, 1, \ldots, q-1$. The quasiparticles in the dimension q representation are of the form $\sigma e^{i(2l+1)\varphi/2\sqrt{q}}$. ψ and σ are the primary fields of the Ising CFT. Consider the fusion rule

$$\sigma e^{i(2l+1)\varphi/2\sqrt{q}} \times \sigma e^{i(2l'+1)\varphi/2\sqrt{q}} = (1 + \psi) e^{i(l+l'+1)\varphi/\sqrt{q}}. \tag{41}$$

The fact that both 1 and ψ are produced and not either one individually can now be seen to be a special case of the analysis above, since $\gcd(1, 1, 2) = 1$, all quasiparticles in the dimension $2q$ representation that have the allowed charge must be produced from the fusion of quasiparticles in the dimension q representation.

C. Fusion rules, domain walls, and pattern of zeros

The pattern of zeros sequences $\{l_{\gamma;a}\}$ defined thus far are interpreted by supposing that there is a quasiparticle V_γ at the origin while electrons are successively brought in toward it. $l_{\gamma;a}$ characterizes the order of the zero that results in the correlation function (i.e., the wave function) as the ath electron is brought in.

Generalize this concept. Imagine putting b electrons at the origin and having a sequence of integers $\{l_{b;a}\}$ that characterizes the order of the zeros as electrons are sequentially brought in to the origin until, after some number a_0 of electrons are brought in, the quasiparticle V_γ is taken to the origin and fused with the electrons there. We then continue to bring additional electrons in and obtain the rest of the sequence. In terms of the quasiparticle sequence $\{l_{\gamma;a}\}$, the combined sequence $\{l_{b,\gamma;a}\}$ would be given by

$$l_{b,\gamma;a} = \begin{cases} l_{b;a} & \text{if } a \leq a_0 \\ l_{\gamma+b;a} & \text{if } a > a_0. \end{cases} \tag{42}$$

If a_0 is large enough, the occupation-number sequence $\{n_l\}$ that corresponds to $\{l_{b,\gamma;a}\}$ will have a domain-wall structure. The first a_0 particles will be described by the sequence $\{n_{b;l}\}$, while the remaining particles will be described by the se-

quence $\{n_{\gamma+b;l}\}$. We see that a quasiparticle not at the origin corresponds to a domain wall between the ground-state occupation distribution $\{n_{b;l}\}$ and the quasiparticle occupation distribution $\{n_{\gamma+b;l}\}$. In the large l limit, $\{n_{\gamma+b;l}\}=\{n_{\gamma;l}\}$ and the asymptotic $\{n_{\gamma;l}\}$ indicates that there is a quasiparticle γ near the origin.

Extending this concept further, we see that upon bringing V_γ in to the origin after a_0 electrons, we can bring another quasiparticle $V_{\gamma'}$ in to the origin after yet another set of—say—a_1 electrons have sequentially been taken to the origin. The sequence $\{l_{\gamma';a}\}$ for $a > a_1$ will describe a new quasiparticle γ'' that can be regarded as a bound state of two quasiparticles γ and γ' near the origin. This suggests that by considering the sequence in such a situation, we can determine the fusion rules of the quasiparticles.

However, the fusion of non-Abelian quasiparticles can be quite complicated, as indicated by the fusion rules,

$$V_\gamma V_{\gamma'} \sim \sum_{\gamma''} N_{\gamma\gamma'}^{\gamma''} V_{\gamma''}, \qquad (43)$$

which suggests that the bound state of quasiparticles γ and γ' can correspond to several different quasiparticles γ''. Can the consideration of the above sequence capture such a possibility of multiple fusion channels?

The answer is yes. Suppose γ and γ' can fuse to γ''. Then, the above consideration of the fusion of quasiparticles γ and γ' will generate a sequence $\{l_{b,\gamma,\gamma';a}\}$,

$$l_{b,\gamma,\gamma';a} = \begin{cases} l_{b;a} & \text{if } a \le a_0 \\ l_{\gamma+b;a} & \text{if } a_0 < a \le a_1 \\ l_{\gamma'+b;a} & \text{if } a > a_1. \end{cases} \qquad (44)$$

The occupation-number sequence in this case will have two domain walls. For the first a_0 particles, the sequence will be described by $\{n_{b;l}\}$ which corresponds to the ground state. For the next a_1 particles it will be described by $\{n_{\gamma;l}\}$, which is a sequence that corresponds to the quasiparticle γ. After a_1 it will be described by $\{n_{\gamma'';l}\}$ which is a sequence that corresponds to the quasiparticle γ''. In this picture, *an occupation-number sequence that contains domain walls separating sequences that belong to different quasiparticles describes a particular fusion channel for several quasiparticles that are fused together.*[25]

In Ref. [25] the fusion rules for parafermion FQH states were obtained from the pattern of zeros by identifying the domain walls that correspond to "elementary" quasiparticles in parafermion FQH states. In the following, we will describe a very different and generic approach that applies to all FQH states described by pattern of zeros.

Notice the absence of the sequence $\{n_{\gamma';l}\}$ in the above consideration of the fusion $\gamma\gamma' \to \gamma''$, even though the quasiparticle γ' was part of the fusion. Here the quasiparticle γ' appears implicitly as a domain wall between $\{n_{\gamma;l}\}$ and $\{n_{\gamma'';l}\}$. This motivates us to view the fusion from a different angle; what quasiparticle can fuse with quasiparticle γ to produce the quasiparticle γ''? From this point of view, we may try to determine $\{n_{\gamma';l}\}$ from $\{n_{\gamma;l}\}$ and $\{n_{\gamma'';l}\}$ to obtain
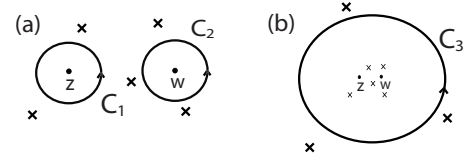

FIG. 1. Schematic drawing of contours around which y is taken. The crosses depict off-particle zeros. (a) y is taken around C_1 and then C_2, not enclosing any off-particle zeros. The correlation function picks up a phase $2\pi(p_{\gamma+a;b}+p_{\gamma'+c;b})$. (b) z and w are brought together, fusing $V_{\gamma+a}(z)$ and $V_{\gamma'+c}(w)$, dragging in some of the off-particle zeros along with them. y is taken around the fused combination, as illustrated by the contour C_3. The correlation function picks up a phase $2\pi p_{\gamma'+a+c;b} \ge 2\pi(p_{\gamma+a;b}+p_{\gamma'+c;b})$.

the fusion rules. Or more generally, the three occupation distributions $\{n_{\gamma;l}\}$, $\{n_{\gamma';l}\}$, and $\{n_{\gamma'';l}\}$ should satisfy certain conditions if γ and γ' can fuse into γ''.

Let us now look for such a condition. Suppose two quasiparticle operators γ and γ' can fuse to a third one γ'' and consider the OPE between the following three operators:

$$V_{\gamma+a}(z)V_{\gamma'+c}(w)V_b(y) \sim f(z,w,y)V_{\gamma'+a+b+c}(z) + \dots \quad (45)$$

(Such an OPE makes sense if we are imagining a correlation function with all other operators inserted at points far away from z, w, and y.) Let us first fix all positions except y and regard the correlation function as a function of y. Zeros (poles) of the correlation function can occur when y coincides with the positions at which other operators are inserted. However, zeros can also occur at locations away from the particles (see Fig. 1). Imagine that we take y around w without enclosing any of the off-particle zeros. The phase that the correlation function acquires upon such a monodromy is simply $2\pi p_{\gamma'+c;b}$. In terms of the scaling dimensions, the integer $p_{\gamma;b}$ is given by

$$p_{\gamma;b} = h_{\gamma+b} - h_\gamma - h_b = p_{\gamma;b}^{\text{sc}} + bQ_\gamma/\nu, \qquad (46)$$

where

$$p_{\gamma;b}^{\text{sc}} = h_{\gamma+b}^{\text{sc}} - h_\gamma^{\text{sc}} - h_b^{\text{sc}} = \sum_{a=1}^{b}(l_{\gamma;a}^{\text{sc}} - l_a^{\text{sc}}). \qquad (47)$$

If we take y around z without enclosing any off-particle zeros, the correlation function acquires a phase $2\pi p_{\gamma+a;b}$. Taking y around w and then around z thus gives a total phase of $2\pi(p_{\gamma+a;b}+p_{\gamma'+c;b})$. Compare that combined process with the following process: fuse $V_{\gamma+a}$ and $V_{\gamma'+c}$ to get $V_{\gamma'+a+c}$ and take $V_b(y)$ around $V_{\gamma'+a+c}$; physically this corresponds to taking z to be close to w compared to y and taking y around a contour that encloses both z and w (Fig. 1). The result of such an operation is that the correlation function acquires a phase of $2\pi p_{\gamma'+a+c;b}$. In fusing $V_{\gamma+a}(z)$ and $V_{\gamma'+c}(w)$ to get $V_{\gamma'+a+c}(z)$, some of the off-particle zeros that were present before the fusion may now be located at z. That is, fusing $V_{\gamma+a}(z)$ and $V_{\gamma'+c}(w)$ corresponds to taking $w \to z$, which in the process may take some of the off-particle zeros to z as well. Therefore we can conclude,

$$p_{\gamma+a;b} + p_{\gamma'+c;b} \le p_{\gamma'+a+c;b}, \qquad (48)$$

which must be satisfied for all positive integers a, b, and c. The inequality is saturated when there are no off-particle zeros at all.

The U(1) Abelian fusion rules imply charge conservation $Q_\gamma + Q_{\gamma'} = Q_{\gamma''}$, which means that the U(1) part saturates the inequality (48) [see Eq. (46)]. This allows us to obtain a more restrictive condition

$$p^{\rm sc}_{\gamma+a;b} + p^{\rm sc}_{\gamma'+c;b} \le p^{\rm sc}_{\gamma'+a+c;b}, \qquad (49)$$

which corresponds to Eq. (48) applied to the simple-current part. In terms of the sequences $\{l_{\gamma,a}\}$, the condition (49) becomes

$$\sum_{j=1}^{b} (l^{\rm sc}_{\gamma+a;j} + l^{\rm sc}_{\gamma'+c;j} - l^{\rm sc}_{j}) \le \sum_{j=1}^{b} l^{\rm sc}_{\gamma'+a+c;j}. \qquad (50)$$

Remember that $l^{\rm sc}_{\gamma;a}$ is obtained from $l_{\gamma,a}$ through Eqs. (23) and (21). The pattern of zeros sequences $\{l_{\gamma,a}\}$ that describe valid quasiparticles are solved from Eq. (13).

For states that satisfy the n-cluster condition, the scaling dimensions and hence the pattern of zeros have a periodicity of n,

$$p^{\rm sc}_{\gamma+a+n;b} = p^{\rm sc}_{\gamma+a;b} = p^{\rm sc}_{\gamma+a;b+n}. \qquad (51)$$

Therefore, a, b, and c need only run through the values $0, \ldots, n-1$.

Equation (49) or Eq. (50) is the condition that we are looking for. The fusion coefficient $N^{\gamma''}_{\gamma\gamma'}$ can be nonzero only if the triplet $(\gamma, \gamma', \gamma'')$ conserves charge $Q_\gamma + Q_{\gamma'} = Q_{\gamma''}$ and satisfies Eq. (49) [or Eq. (50)] for any choice of a, b, c. This result allows us to calculate the fusion rules from the pattern of zeros.

Remarkably, the condition (49) or Eq. (50) appears to be complete enough. We find through numerical tests that for the generalized and composite parafermion states discussed below, condition (49) is sufficient to obtain the fusion rules: V_γ, $V_{\gamma'}$, and $V_{\gamma''}$ satisfy Eq. (49) and charge conservation if and only if V_γ and $V_{\gamma'}$ can fuse to give $V_{\gamma''}$. We do not yet know, aside from these parafermion states, whether condition (49) is sufficient to obtain the fusion rules. On the other hand, if we assume $N^{\gamma''}_{\gamma\gamma'} = 0, 1$ then Eq. (49) or Eq. (50) completely determines the fusion rules.

D. Fusion rules and ground-state degeneracy on genus g surfaces

After obtaining the fusion rules from the pattern of zeros [see Eq. (49)], we like to ask: can we check this result physically? (Say through numerical calculations). Given a pattern of zeros sequence, there is a local Hamiltonian, which was constructed in Ref. 14, whose ground-state wave function is described by this pattern of zeros. The Hamiltonian can be solved numerically to obtain quasiparticle excitations and, in principle, we can check the fusion rules. However, this approach does not really work since the numerical calculation will produce many quasiparticle excitations and most of
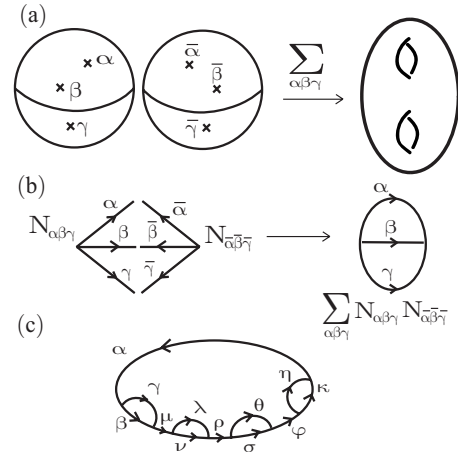

FIG. 2. (a) Genus g surfaces can be constructed by sewing together three-punctured spheres; the case $g=2$ is shown here. (b) Three-punctured spheres can be depicted by vertices of φ^3 diagrams. There is a factor of $N_{\alpha\beta\gamma}$ for each vertex with outward directed edges α, β, and γ. Reversing the direction of an edge corresponds to replacing a quasiparticle with its conjugate. The $g=2$ case is depicted by a two-loop diagram, and has a factor $\sum_{\alpha\beta\gamma} N_{\alpha\beta\gamma} N_{\bar\alpha\bar\beta\bar\gamma}$. (c) An example of a five-loop diagram, corresponding to a genus $g=5$ surface. This would give a factor $\sum N_{\bar\alpha\beta\gamma} N_{\bar\beta\bar\gamma\mu} N_{\bar\mu\nu\lambda} N_{\bar\lambda\bar\nu\rho} N_{\bar\rho\theta\sigma} N_{\bar\theta\bar\sigma\varphi} N_{\bar\varphi\eta\kappa} N_{\bar\kappa\bar\eta\alpha}$, which can be written more compactly as $\mathrm{Tr}(\sum_\alpha N_\alpha N_{\bar\alpha})^{g-1}$.

them only differ by local excitations and should be regarded as equivalent. We do not have a good way to determine which quasiparticles are equivalent and which are topologically distinct. This is why we cannot directly check the fusion rules through the excitations obtained from numerical calculations.

However, there is an indirect way to check the fusion rules. The fusion rules in a topological phase also determine the ground-state degeneracy on genus g surfaces. We can numerically compute the ground-state degeneracy on a genus g surface and compare it with the result from the fusion rule.

Why do the fusion rules determine the ground-state degeneracy? This is because genus g surfaces may be constructed by sewing together three-punctured spheres (see Fig. 2). Each puncture is labeled by a quasiparticle type and two punctures can be sewed together by summing over intermediate states at the punctures. This corresponds to labeling one puncture by a quasiparticle γ, labeling the other puncture by the conjugate of γ, which is referred to as $\bar\gamma$, and summing over γ. $\bar\gamma$ is the unique quasiparticle that satisfies $N^0_{\gamma\bar\gamma}=1$; the operator that takes γ to $\bar\gamma$ is the charge-conjugation operator $C: C_{\alpha\beta}=N^0_{\alpha\beta}$. The dimension of the space of states of a three-punctured sphere labeled by α, β, and γ is $N_{\alpha\beta\gamma}=N^{\bar\gamma}_{\alpha\beta}$. $N_{\alpha\beta\gamma}$ is symmetric in its indices, which we can raise and lower with the charge-conjugation operator,

$$N_{\alpha\beta\gamma} = C_{\gamma\delta} N^\delta_{\alpha\beta} = N^{\bar\gamma}_{\alpha\beta} = C^{\bar\gamma\delta} N_{\alpha\beta\delta}.$$

$C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$: $C^{\alpha\beta} C_{\beta\gamma} = \delta^\alpha_\gamma$. Also, note that C squares to the identity $C_{\alpha\beta} C_{\beta\gamma} = \delta_{\alpha\gamma}$, so that C is its own inverse $C_{\alpha\beta} = C^{\alpha\beta}$. If we represent a three-punctured sphere by a vertex in a φ^3 diagram with directed edges and label the

outgoing edges by α, β, and γ, each vertex comes with a factor $N_{\alpha\beta\gamma}$. A genus g surface can then be thought of as a g-loop diagram. This implies that the ground-state degeneracy on a torus, for example, is $\Sigma_{\alpha,\beta} N_{0\alpha\beta} N_{0\bar\alpha\bar\beta}$. The ground-state degeneracy on a genus 2 surface would be given by

$$\sum_{\alpha\beta\gamma} N_{\alpha\beta\gamma} N_{\bar\alpha\bar\beta\bar\gamma} = \sum_{\alpha\beta\gamma} N_{\alpha\beta}^{\gamma} N_{\bar\alpha\gamma}^{\beta}.$$

In general, one obtains the following formula for the ground-state degeneracy in terms of the fusion rules[26] (see Fig. 2):

$$\text{GSD} = \text{Tr}\left(\sum_{a=0}^{N-1} N_\alpha N_{\bar\alpha}\right)^{g-1}. \tag{52}$$

N is the number of quasiparticle types, $(N_\alpha)_\beta^\gamma = N_{\alpha\beta}^\gamma$, and matrix multiplication of the fusion matrices is defined by contracting indices, so that $(N_\alpha N_\beta)_i^j = N_{\alpha i}^k N_{\beta k}^j$. Equation (52) assumes that all fields are fusing to the identity, so it applies only when the total number of electrons is a multiple of n (for n-cluster states). For other cases, one must perform a more careful analysis.

We show in Appendix B that Eq. (52) can be rewritten as

$$\text{GSD} = \left(\sum_{\gamma=0} d_\gamma^2\right)^{g-1} \sum_{\gamma=0}^{N-1} d_\gamma^{-2(g-1)}. \tag{53}$$

d_γ is the "quantum dimension" of quasiparticle γ. It is given by the largest eigenvalue of the fusion matrix N_γ, and it has the property that the space of states with n quasiparticles of type γ at fixed locations goes as $\sim d_\gamma^n$ for large n. In particular, Abelian quasiparticles have unit quantum dimension. It is remarkable that the ground-state degeneracy on any surface is determined solely by the quantum dimensions of quasiparticles.

From Eq. (53) and the magnetic algebra structure of the quasiparticles, we can prove that the ground-state degeneracy on genus g surfaces factorizes into a part that depends only on the filling fraction ν and a part that depends only on the simple-current CFT. In particular, we show in Eq. (A4) that Eq. (53) can be rewritten as

$$\text{GSD} = \nu^{-g}\left(\sum_i c_i^{sc} d_i^2\right)^{g-1}\left(\sum_i c_i^{sc} d_i^{-2(g-1)}\right). \tag{54}$$

Here Σ_i sums over the representations (which are labeled by i) of the magnetic translation algebra. d_i is the quantum dimension of quasiparticles in the ith representation. c_i^{sc} is the number of distinct fields of the form $\psi^a \sigma_i$ for a fixed i. It can be determined from the pattern of zeros as follows. Recall that all of the quasiparticles in the ith representation of the magnetic translation algebra have the same sequence $\{l_{i;a}^{sc}\}$ up to a cyclic permutation. c_i^{sc} describes the shortest period of $\{l_{i;a}^{sc}\}$,

$$l_{i;a}^{sc} = l_{i;a+c_i^{sc}}^{sc}.$$

$\{l_{i;a}^{sc}\}$ always satisfies $l_{i;a}^{sc} = l_{i;a+n}^{sc}$ and very often $c_i^{sc} = n$. But sometimes, c_i^{sc} can be a factor of n.

We see that c_i^{sc} is determined from the pattern of zeros of the quasiparticles. We have seen that (under certain assumptions) the fusion rules (and hence the quantum dimensions d_i) can also be determined from the pattern of zeros. Thus Eq. (54) allows us to calculate the ground-state degeneracy on any genus g surface from the pattern of zeros.

Equation (54) shows that the ground-state degeneracy on a genus g surface factorizes into ν^{-g} times a factor that depends only on the simple-current CFT. This is remarkable because ν^{-1} is generically not an integer. The second factor may be interpreted as the dimension of the space of conformal blocks on a genus g surface with no punctures for the simple-current CFT. In particular, for genus one, this gives $1/\nu$ times the number of distinct fields of the form $\psi^a \sigma_i$ in the simple-current CFT; a result which we find more explicitly in Sec. IV for the parafermion quantum Hall states. Note that this formula assumes that the number of electrons is a multiple of n; we expect a similar decomposition into ν^{-g} times a factor that depends only on the simple-current CFT if the electron number is not a multiple of n, but we will not analyze here this more complicated case.

IV. PARAFERMION QUANTUM HALL STATES

Using the pattern of zeros approach, we can obtain the number of types of quasiparticles,[14,15] the fusion rules, etc. However, to obtain those results from the pattern of zeros approach, we have made certain assumptions. In this section, we will study some FQH states using the CFT approach to confirm those results obtained from the pattern of zeros approach.

The quantum Hall states to which we now turn include the parafermion,[18] "generalized parafermion," and "composite parafermion"[14] states. These states are all based on the Z_n parafermion conformal field theory introduced by Zamolodchikov and Fateev.[20] In the context of quantum Hall states, we focus on the holomorphic part of the theory and leave out the antiholomorphic part. The Z_n parafermion CFT is generated by n simple currents $\psi_a(z)$, $a = 0, \ldots, n-1$, which have a Z_n symmetry $\psi_1^n = 1$ and $\psi_a = \psi_1^a$.

The field space of the theory splits into a direct sum of subspaces, each with a certain Z_n charge, labeled by l with $l = 0, \ldots, n-1$. The fields with minimal scaling dimension in each of these subspaces are the so-called "spin fields" or "disorder operators" σ_l. Fields in each subspace are generated from the σ_l by acting with the simple currents $\psi_a \sigma_l$. Based on a relation between SU(2) current algebra and parafermion theory, a way of labeling these primary fields is as Φ_m^l.[21] The spin fields are $\sigma_l = \Phi_l^l$ and the simple currents are $\psi_a = \Phi_{2a}^0$. The Z_n symmetry implies that $\Phi_{m+2n}^l = \psi_1^n \Phi_m^l = \Phi_m^l$. The scaling dimensions of the simple currents $\psi_a = \Phi_{2a}^0$ are chosen to be

$$\Delta_{2a}^0 = \frac{a(n-a)}{n}. \tag{55}$$

Such a choice then determines the scaling dimensions of the rest of the fields in the theory. The scaling dimension Δ_m^l of the field Φ_m^l is given by
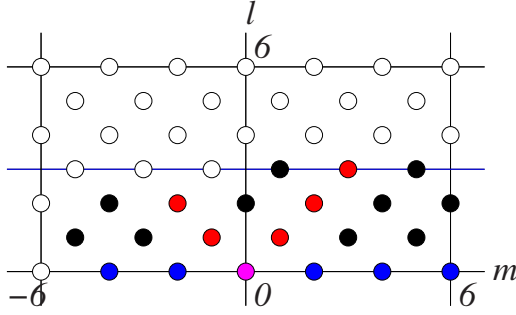
FIG. 3. (Color online) The filled dots represent the distinct Φ_m^l's in Z_6 parafermion CFT. The blue dots (dots for which $l=0$) represent the simple-current operators $\Phi_{2a}^0 = \psi_a$ and the red dots (dots for which $l=m$) represent the spin operators $\Phi_l^l = \sigma_l$.

$$\Delta_m^l = \begin{cases} \dfrac{l(l+2)}{4(n+2)} - \dfrac{m^2}{4n} + \dfrac{m-l}{2} & \text{if } l \leq m \leq 2n - l \\[2ex] \dfrac{l(l+2)}{4(n+2)} - \dfrac{m^2}{4n} & \text{if } -l \leq m \leq l. \end{cases} \tag{56}$$

l and m satisfy

$$l + m = \text{even}, \quad 0 \leq l \leq n. \tag{57}$$

In the $\widehat{su}(2)_n/\hat{u}(1)$ coset formulation of the parafermion CFTs, the following field identifications are made:

$$\{l, m\} \sim \{n - l, m - n\} \sim \{n - l, n + m\}. \tag{58}$$

Also, the Z_n structure implies

$$\{l, m\} \sim \{l, m + 2n\}. \tag{59}$$

From Eqs. (57)–(59), we can find the distinct Φ_m^l's (see Fig. 3). In the pattern of zeros picture, the identification (58) is natural because quasiparticles containing the field Φ_m^l yield the same pattern of zeros as those containing the field Φ_{n+m}^{n-l} and for this reason are considered to be physically equivalent.

The electron operator in the conventional Read-Rezayi Z_n parafermion quantum Hall states at filling fraction ν is given by

$$V_e(z) = \psi_1(z) e^{i\sqrt{1/\nu}\varphi(z)}. \tag{60}$$

In the "generalized" $Z_n^{(k)}$ parafermion states, we use ψ_k instead of ψ_1 to define the electron operator,

$$V_e(z) = \psi_k(z) e^{i\sqrt{1/\nu}\varphi(z)}. \tag{61}$$

The states $Z_n^{(1)}$ thus correspond to the conventional Read-Rezayi states. The condition that the electron operator have integer or half-integer spin translates into a discrete set of possible filling fractions for the $Z_n^{(k)}$ parafermion states,

$$\nu = \frac{n}{nM + 2k^2}. \tag{62}$$

M is a non-negative integer; that it must be nonnegative is derived from the condition that the OPE between two electrons must not diverge as two electrons are brought close to

each other. Equation (62) is the generalization of the well-known formula $\nu = \frac{n}{nM+2}$ for the conventional Z_n Read-Rezayi parafermion states. In what follows, we assume k and n are coprime; cases in which they are not must be treated differently.

The quasiparticle fields take the form

$$V_\gamma = \Phi_m^l e^{iQ_\gamma\sqrt{1/\nu}\varphi}, \tag{63}$$

where Q_γ is the electric charge of the quasiparticle. V_γ is a valid quasiparticle if and only if it has a single-valued OPE with the electron operator. To find the number of distinct quasiparticle types, we need to find all the valid quasiparticle operators V_γ while regarding two quasiparticle operators as equivalent if they differ by an electron operator.

Since quasiparticle operators that differ by an electron operator are regarded as equivalent, every quasiparticle is equivalent to one whose charge lies between 0 and 1. Thus a simple way of dealing with this equivalence relation is to restrict ourselves to considering operators whose charges Q_γ satisfy

$$0 \leq Q_\gamma < 1. \tag{64}$$

This ensures that we consider a single member of each equivalence class because adding an electron to a quasiparticle operator increases its charge by one. For each primary field labeled by $\{l, m\}$, there are only a few choices of Q_γ that satisfy Eq. (64) and that will make the operator V_γ local with respect to the electron operator. Finding all these different allowed charges for each $\{l, m\}$ will give us all the different quasiparticle types.

The OPE between the quasiparticle operator and the electron is

$$V_\gamma(z) V_e(w) \sim (z - w)^a \Phi_{m+2k}^l e^{i(Q_\gamma + 1)1/\sqrt{\nu}\varphi}, \tag{65}$$

$$a = \Delta_{m+2k}^l - \Delta_m^l - \Delta_{2k}^0 + Q_\gamma/\nu. \tag{66}$$

Locality (single valuedness) between the quasiparticle and the electron implies that a must be an integer. Each allowed charge Q_γ for a given primary field $\{l, m\}$ can therefore be labeled by an integer a that, from Eq. (64), satisfies

$$0 \leq a - \Delta_{m+2k}^l + \Delta_m^l + \Delta_{2k}^0 < 1/\nu. \tag{67}$$

Therefore, to find all the distinct valid quasiparticles, we search through all of the distinct, allowed triplets $\{a, l, m\}$, subject to Eq. (57) and the identifications in Eqs. (58) and (59), and find those that satisfy Eq. (67). Carrying out this program on a computer, we learn that the number of quasiparticles in the generalized $Z_n^{(k)}$ parafermion states follows the formula:

$$\text{No. of quasiparticles} = \frac{1}{2}\frac{n(n+1)}{\nu}. \tag{68}$$

This is the natural generalization of the formula $\frac{1}{2}(nM + 2)(n+1)$ that is well known for the $k=1$ case.

The above approach has yielded not only these generalized parafermion states but also a series of "composite parafermion" states. In these states, the relevant conformal field theory is chosen to consist of several parafermion conformal

TABLE II. Pattern of zeros, scaling dimensions, and charges for the quasiparticles in the $Z_3^{(1)}$ state at ν $=3/2$ (for bosonic electrons). The periodic sequence $\tilde{l}_{\gamma;a}^{sc}$ is listed for $a=1,\ldots,3$. The asymptotic form of a single unit cell of $n_{\gamma;l}$ is shown.

	$Z_3^{(1)}$ $\nu=3/2$				
$\{Q_\gamma;l,m\}$	$n_{\gamma;l}$	$n\tilde{l}_{\gamma;a}^{sc}$	h_γ	h_γ^{sc}	$h_{\gamma,min}^{sc}$
$\{0;\,0,\,0\}$	30	2 0 −2	0	0	0
$\{1/2;\,0,\,-2\}$	03	−2 2 0	3/4	2/3	0
$\{1/2;\,1,\,1\}$	21	1 −1 0	3/20	1/15	1/15
$\{0;\,1,\,3\}$	12	−1 0 1	2/15	2/5	1/15

field theories taken together of the form $\otimes_i Z_{n_i}^{(k_i)}$. We emphasize that here $\{n_i\}$ are all coprime with respect to one another and k_i is coprime with respect to n_i. Cases in which these coprime conditions do not hold should be treated differently. Here, the electron operator is

$$V_e = \prod_{i=1} \psi_{k_i;n_i} e^{i\sqrt{1/\nu}\varphi}, \qquad (69)$$

where $\psi_{k_i;n_i}$ is a simple current of the Z_{n_i} parafermion CFT. The condition for the filling fraction [Eq. (62)] generalizes to

$$\nu = \frac{N}{NM + 2N\sum_i \frac{k_i^2}{n_i}}, \qquad (70)$$

where $N=\prod_i n_i$ and M is a non-negative integer. Following a procedure similar to that described above in the generalized parafermion case, the condition (67) generalizes to

$$0 \leq a - \sum_i (\Delta_{m_i+2k_i;n_i}^{l_i} + \Delta_{m_i;n_i}^{l_i} + \Delta_{2k_i;n_i}^{0}) < 1/\nu, \qquad (71)$$

where $\Delta_{m_i;n_i}^{l_i}$ is the scaling dimension $\Delta_{m_i}^{l_i}$ from the Z_{n_i} parafermion CFT. We find that the number of quasiparticles follows the natural generalization of Eq. (68):

$$\text{No. of quasiparticles} = \frac{1}{\nu}\prod_i \frac{n_i(n_i+1)}{2}. \qquad (72)$$

Strikingly, these results agree with the number of quasiparticles computed in an entirely different fashion through the pattern of zeros approach.[15]

Since we know the fusion rules in the Z_n parafermion CFTs, we can easily examine the fusion rules in the parafermion quantum Hall states. The most general states that we have discussed in this section have been the $\otimes_i Z_{n_i}^{(k_i)}$ composite parafermion states. The quasiparticles operators can be written as

$$V_\gamma = \prod_{i=1} \Phi_{m_i;n_i}^{l_i} e^{iQ_\gamma\sqrt{1/\nu}\varphi}, \qquad (73)$$

where $\Phi_{m_i;n_i}^{l_i}$ is the primary field $\Phi_{m_i}^{l_i}$ from the Z_{n_i} parafermion CFT. Equivalently, we can label each quasiparticle as $\{Q_\gamma;l_1,m_1,l_2,m_2,\cdots\}$.

The primary fields Φ_m^l in the Z_n parafermion CFT have the following fusion rules:[21]

$$\Phi_m^l \times \Phi_{m'}^{l'} = \sum_{j=|l-l'|}^{\min(l+l',2n-l-l')} \Phi_{m+m'}^j. \qquad (74)$$

Therefore, in terms of the $\{Q_\gamma;l_1,m_1,\cdots\}$ labels, the fusion rules for the quasiparticles in the composite parafermion states are given by

$$\{Q_\gamma;l_1,m_1,\cdots\} \times \{Q_{\gamma'};l_1',m_1',\cdots\}$$

$$= \sum_{l_i''=|l_i-l_i'|}^{\min(l_i+l_i',2n_i-l_i-l_i')} \{Q_\gamma + Q_{\gamma'};l_1'',m_1+m_1',l_2'',m_2+m_2',\cdots\},$$

$$\qquad (75)$$

where there is a sum over each l_i'' for $i=1,2,\ldots,$ and we make the identifications

$$\{Q_\gamma;\cdots,l_i,m_i,\cdots\} \sim \{Q_\gamma;\cdots,n_i-l_i,m_i-n_i,\cdots\}$$

$$\sim \{Q_\gamma;\cdots,l_i,m_i+2n_i,\cdots\}$$

$$\sim \{Q_\gamma+1;\cdots,l_i,m_i+2k_i,\cdots\}. \qquad (76)$$

These fusion rules agree with that obtained previously from the pattern of zeros using Eqs. (49) or (50).

V. EXAMPLES

Now we will describe some specific examples of the parafermion states, listing their pattern of zeros, scaling dimensions, ground-state degeneracies, and discussing their fusion rules.

In the $Z_3^{(1)}$ state at $\nu=3/2$ (which is the bosonic Z_3 parafermion state[18]), Table II shows that there are two representations of the magnetic algebra, with two quasiparticles in each representation. These two representations are irreducible ($q=m$), so the fusion rules decompose as Eq. (39). Labeling these two by 1 (the identity) and σ, we see that they satisfy the fusion rules

$$\sigma\sigma = 1 + \sigma. \qquad (77)$$

There are only two modular tensor categories of rank 2, the so-called semion modular tensor category (MTC) and the

TABLE III. Pattern of zeros, scaling dimensions, and charges for the quasiparticles in the $Z_5^{(1)}$ state at $\nu=5/2$ (for bosonic electrons). The periodic sequence $\tilde{l}_{\gamma,a}^{\mathrm{sc}}$ is listed for $a=1,\ldots,5$. The asymptotic form of a single unit cell of $\{n_{\gamma,l}\}$ is shown.

		$Z_5^{(1)}$ $\nu=5/2$			
$\{Q_\gamma;l,m\}$	$n_{\gamma;l}$	$n\tilde{l}_{\gamma,a}^{\mathrm{sc}}$	h_γ	h_γ^{sc}	$h_{\gamma,\mathrm{min}}^{\mathrm{sc}}$
$\{1/2;0,6\}$	05	-2 -4 4 2 0	$\frac{5}{4}$	$\frac{6}{5}$	0
$\{0;0,0\}$	50	4 2 0 -2 -4	0	0	0
$\{1/2;1,1\}$	41	3 1 -1 -3 0	$\frac{3}{28}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{0;1,5\}$	14	-1 -3 0 3 1	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{2}{35}$
$\{1/2;2,6\}$	23	-2 1 -1 2 0	$\frac{15}{28}$	$\frac{17}{35}$	$\frac{3}{35}$
$\{0;2,0\}$	32	-1 2 0 -2 1	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{3}{35}$

Fibonacci MTC, and we see that these fusion rules correspond to the Fibonacci MTC.[27]

In the $Z_5^{(k)}$ states,[14] we also have $q=m$ and the quasiparticles form three irreducible representations of the magnetic algebra. The fusion algebra again has the simple decomposition Eq. (39). In the $Z_5^{(1)}$ state at $\nu=5/2$, Table III shows that there are two quasiparticles in each irreducible representation, while in the $Z_5^{(2)}$ state at $\nu=5/8$, Table IV shows that there are eight quasiparticles in each irreducible representation of the magnetic algebra. The nontrivial non-Abelian part of the fusion rules is given by the fusion rules among the three irreducible representations. Labeling these three by 1, σ_1, and σ_2, and using Eq. (49) or Eq. (75), we can see that they satisfy the following fusion rules:

$$\sigma_1\sigma_1 = 1 + \sigma_2,$$

$$\sigma_2\sigma_2 = 1 + \sigma_1 + \sigma_2,$$

$$\sigma_1\sigma_2 = \sigma_1 + \sigma_2. \tag{78}$$

This corresponds to the $(A_1,5)_{1/2}$ MTC described in Ref. 27.

We see that when $q=m$, the decomposition of the fusion rules (39) into a nontrivial non-Abelian part that depends only on how the different irreducible representations fuse together and a trivial Abelian part greatly simplifies these states. The $Z_3^{(1)}$ state, which at $\nu=3/2$ contains four quasiparticles, has only two irreducible representations of the magnetic algebra and therefore the non-Abelian part is described

by a simple rank 2 MTC (Ref. 27): the Fibonacci MTC. Similarly, $Z_5^{(k)}$, which for $k=2$ and $\nu=5/8$ has 24 quasiparticles, actually has only three different irreducible representations of the magnetic algebra, and therefore the non-Abelian part of its fusion rules is described by a simple rank 3 MTC. The Z_2 states listed previously in Table I have two representations, yet one of them is not irreducible. It turns out that the nontrivial non-Abelian part of the fusion rules in the Z_2 states is described by the rank 3 Ising MTC. So, even though $Z_5^{(2)}$ at first sight seems a great deal more complicated than Z_2, their non-Abelian parts have the same degree of complexity, namely, they are both described by a rank 3 MTC.

Using the fusion rules (75), we can also verify that the ground-state degeneracy on genus g surfaces follows the decomposition (54). In particular, for the Z_n parafermion CFTs of Zamolodchikov and Fateev, the quantum dimensions of the fields $\sigma_l \equiv \Phi_l^l$ can be found from the relation of these theories to $SU(2)_k$ Wess-Zumino-Witten (WZW) models.[28] The result is

$$d_l = \frac{\sin\left(\frac{\pi(l+1)}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}. \tag{79}$$

From the relation (58), it follows that for n even, there are $\frac{n}{2}+1$ distinct σ_l's. $c_l^{\mathrm{sc}}=n$ for $l=0,\ldots,n/2-1$ and $c_{n/2}^{\mathrm{sc}}=n/2$. For n odd, there are $\frac{n+1}{2}$ distinct σ_l's, and $c_l^{\mathrm{sc}}=n$ for $l=0,\cdots,\frac{n-1}{2}$. Using Eq. (54), we find that the ground-state degeneracy for the $Z_n^{(k)}$ states on a genus g surface is given by

$$\text{GSD} = \nu^{-g}n^g \times \begin{cases} \left[\sum_{l=1}^{n/2}\sin^2\left(\frac{l\pi}{n+2}\right) + \frac{1}{2}\right]^{g-1}\left\{\sum_{l=1}^{n/2}\left[\sin\left(\frac{l\pi}{n+2}\right)\right]^{-2(g-1)} + \frac{1}{2}\right\} & \text{if } n \text{ is even} \\ \left[\sum_{l=1}^{(n+1)/2}\sin^2\left(\frac{l\pi}{n+2}\right)\right]^{g-1}\left\{\sum_{l=1}^{(n+1)/2}\left[\sin\left(\frac{l\pi}{n+2}\right)\right]^{-2(g-1)}\right\} & \text{if } n \text{ is odd.} \end{cases} \tag{80}$$

TABLE IV. Pattern of zeros, scaling dimensions, and charges for the quasiparticles in the $Z_5^{(2)}$ state at $\nu=5/8$ (for bosonic electrons). The periodic sequence $\tilde{l}_{\gamma,a}^{sc}$ is listed for $a=1,\ldots,5$. The asymptotic form of a single unit cell of $\{n_{\gamma,l}\}$ is shown. Note that the charges, modulo 1, of two quasiparticles that are related by a translation $\hat{T}_1$ differ by $\nu=5/8$, as explained in Sec. IIIB.

$\{Q_\gamma; l, m\}$	$n_{\gamma,l}$	$n\tilde{l}_{\gamma,a}^{sc}$	h_γ	h_γ^{sc}	$h_{\gamma,min}^{sc}$
		$Z_5^{(2)}$ $\nu=5/8$			
$\{0;0,0\}$	20102000	6 −2 0 2 −6	0	0	0
$\{5/8;0,0\}$	02010200	6 −2 0 2 −6	$\frac{5}{16}$	0	0
$\{2/8;0,6\}$	00201020	−6 6 −2 0 2	$\frac{5}{4}$	$\frac{6}{5}$	0
$\{7/8;0,6\}$	00020102	−6 6 −2 0 2	$\frac{29}{16}$	$\frac{6}{5}$	0
$\{4/8;0,2\}$	20002010	2 −6 6 −2 0	1	$\frac{4}{5}$	0
$\{1/8;0,8\}$	02000201	0 2 −6 6 −2	$\frac{13}{16}$	$\frac{4}{5}$	0
$\{6/8;0,8\}$	10200020	0 2 −6 6 −2	$\frac{5}{4}$	$\frac{4}{5}$	0
$\{3/8;0,4\}$	01020002	−2 0 2 −6 6	$\frac{21}{16}$	$\frac{6}{5}$	0
$\{0; 1, 5\}$	01110020	−4 3 0 −3 4	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{2}{35}$
$\{5/8; 1, 5\}$	00111002	−4 3 0 −3 4	$\frac{131}{112}$	$\frac{6}{7}$	$\frac{2}{35}$
$\{2/8; 1, 1\}$	20011100	4 −4 3 0 −3	$\frac{3}{28}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{7/8; 1, 1\}$	02001110	4 −4 3 0 −3	$\frac{75}{112}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{4/8; 1, 7\}$	00200111	−3 4 −4 3 0	$\frac{6}{7}$	$\frac{23}{35}$	$\frac{2}{35}$
$\{1/8; 1, 3\}$	10020011	0 −3 4 −4 3	$\frac{75}{112}$	$\frac{23}{35}$	$\frac{2}{35}$
$\{6/8; 1, 3\}$	11002001	0 −3 4 −4 3	$\frac{31}{28}$	$\frac{23}{35}$	$\frac{2}{35}$
$\{3/8;1,−1\}$	11100200	3 0 −3 4 −4	$\frac{19}{112}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{0; 2, 0\}$	10101101	1 −2 0 2 −1	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{3}{35}$
$\{5/8; 2, 0\}$	11010110	1 −2 0 2 −1	$\frac{67}{112}$	$\frac{2}{7}$	$\frac{3}{35}$
$\{2/8; 2, 6\}$	01101011	−1 1 −2 0 2	$\frac{15}{28}$	$\frac{17}{35}$	$\frac{3}{35}$
$\{7/8; 2, 6\}$	10110101	−1 1 −2 0 2	$\frac{615}{560}$	$\frac{17}{35}$	$\frac{3}{35}$
$\{4/8; 2, 2\}$	11011010	2 −1 1 −2 0	$\frac{2}{7}$	$\frac{3}{35}$	$\frac{3}{35}$
$\{1/8;2,−2\}$	01101101	0 2 −1 1 −2	$\frac{11}{112}$	$\frac{3}{35}$	$\frac{3}{35}$
$\{6/8;2,−2\}$	10110110	0 2 −1 1 −2	$\frac{15}{28}$	$\frac{3}{35}$	$\frac{3}{35}$
$\{3/8; 2, 4\}$	01011011	−2 0 2 −1 1	$\frac{67}{112}$	$\frac{17}{35}$	$\frac{3}{35}$

Table V lists the ground-state degeneracies obtained from the above formula for the cases $n=2$,[29] 3, and 4.[25] These are the same results that one would obtain by numerically computing Eq. (52) for the fusion rule (75). For the $Z_n^{(1)}$ states, they also match the same results obtained in Ref. 25.

VI. SUMMARY

Motivated by the characterization of symmetric polynomials and FQH states through the pattern of zeros,[14] we examined the CFT generated by simple currents in terms of the pattern of zeros. This reveals a deep connection between the simple-current CFT and FQH states. The point of view from the pattern of zeros reveals a magnetic translation algebra that acts on the quasiparticles. This allows us to greatly simplify the fusion algebra of the quasiparticles. It also allows us

to show in general that the ground-state degeneracy on genus g surfaces is ν^{-g} times a factor that depends only on the simple-current part of the CFT. More importantly, we are able to derive a necessary condition on the fusion rules of the

TABLE V. Ground-state degeneracies on genus g surfaces for $Z_n^{(k)}$ parafermion quantum Hall states for the case when the number of electrons is a multiple of n. $\varphi=\frac{1+\sqrt{5}}{2}$ is the golden ratio.

	Ground-state degeneracies for $Z_n^{(k)}$ states
n	GSD
2	$\nu^{-g}2^{g-1}(2^g+1)$
3	$\nu^{-g}3^g(1+\varphi^2)^{g-1}(1+\varphi^{-2(g-1)})$
4	$\nu^{-g}2^{g-1}[3^g+1+(2^{2g}-1)(3^{g-1}+1)]$

quasiparticles based on the pattern of zeros. Such a necessary condition is sufficient to produce a full set of fusion rules for quasiparticles if we assume $N_{\gamma\gamma'}^{\gamma''}=0,1$.

The results obtained from the pattern of zeros approach are checked against the known results from the generalized and composite parafermion CFT. In particular, we find that the number of quasiparticles obtained from the pattern of zeros approach[15] agrees with that obtained from the CFT approach. The fusion rules obtained from the pattern of zeros also agree with the result from the CFT calculation for generalized and composite parafermion FQH states. Those agreements further demonstrate that the pattern of zeros approach is quite a powerful tool to characterize and to calculate the topological properties of generic FQH states.

ACKNOWLEDGMENTS

APPENDIX A: SCALING DIMENSIONS OF QUASIPARTICLES

The way the magnetic algebra structure of the quasiparticles factorizes in the fusion rules is also seen in another topological property of the quasiparticles: the scaling dimensions, or spins, of the quasiparticles. Since quasiparticles that belong to the same representation of the magnetic algebra are described by sequences $\{l_{\gamma,a}^{sc}\}$ that are related by a cyclic permutation, from Eq. (25) we see that for each irreducible representation there is a single number $h_{\gamma;\min}^{sc}$ that we need in order to calculate the scaling dimensions of all other quasiparticles in the same representation. $h_{\gamma;\min}^{sc}$ is the minimum of $h_{\gamma+a}^{sc}$ over all the quasiparticles that belong to this representation. Given $h_{\gamma;\min}^{sc}$, the scaling dimension of the quasiparticle $V_{\gamma+a}$ can be calculated from its pattern of zeros $\{\tilde{l}_{\gamma+a;b}^{sc}\}$.

It is not obvious that the information to obtain $h_{\gamma;\min}^{sc}$ for each irreducible representation is even contained in the pattern of zeros. It may be that $\{l_{\gamma,a}\}$ is not enough information to uniquely specify the CFT and therefore also not enough information to completely determine the scaling dimensions of the fields that are contained in the theory. However, in the case where the pattern of zeros corresponds to the (generalized and composite) parafermion states discussed above, there are explicit formulas in terms of $l_{\gamma,a}$ that yield $h_{\gamma;\min}^{sc}$. This comes as no surprise because in these cases, the pattern of zeros completely specifies the CFT. We do not have a formula that can even be applied in the more general situations.

Let us now describe how to calculate the scaling dimension $h_{\gamma+a}$ of the operator $V_{\gamma+a}$ given $h_{\gamma;\min}^{sc}$. First we must find the index a_0 at which

$$h_{\gamma;\min}^{sc} = h_{\gamma+a+a_0}^{sc}. \tag{A1}$$

This is equivalent to finding the index a_0 at which

$$h_{\gamma+k} - h_{\gamma+a+a_0} \geq 0 \tag{A2}$$

for all k because $h_{\gamma;\min}^{sc}$ is defined to be the minimum of $h_{\gamma+k}$ over all k. Recalling that $\tilde{l}_{\gamma+a;b+1}^{sc}=h_{\gamma+a+b+1}^{sc}-h_{\gamma+a+b}^{sc}$, we see that a_0 is therefore the index at which

$$\sum_{i=1}^{k} \tilde{l}_{\gamma+a;a_0+i}^{sc} = h_{\gamma+a+a_0+k}^{sc} - h_{\gamma+a+a_0}^{sc} = h_{\gamma+a+a_0+k}^{sc} - h_{\gamma;\min}^{sc} \geq 0 \tag{A3}$$

for all k. Using Eq. (A3), we can determine a_0 from $\{\tilde{l}_{\gamma,a}^{sc}\}$, after which we can determine $h_{\gamma+a}$ using Eq. (25),

$$h_{\gamma+a}^{sc} = h_{\gamma;\min}^{sc} - \sum_{b=1}^{a_0} \tilde{l}_{\gamma+a;b}^{sc}. \tag{A4}$$

APPENDIX B: GROUND-STATE DEGENERACY ON GENUS g SURFACES

Here we illustrate how Eqs. (53) and (54) can be determined from Eq. (52). First we observe that the fusion matrices N_α commute (and can therefore be simultaneously diagonalized) because the fusion of any three quasiparticles α, β, and γ should be independent of the order in which they are fused together. Remarkably, there is a symmetric unitary matrix S known as the modular S matrix, which squares to the charge-conjugation operator $S_{\alpha\beta}S_{\beta\gamma}=C_{\alpha\gamma}$ and that simultaneously diagonalizes all of the fusion matrices,[26,30]

$$N_{\alpha\beta}^\gamma = \sum_n S_{\beta n}\lambda_\alpha^{(n)}S_{\gamma n}^\dagger. \tag{B1}$$

Using Eq. (B1) and the fact that $N_{\alpha 0}^\beta=\delta_\alpha^\beta$, the eigenvalues $\lambda_\alpha^{(n)}$ of the fusion matrix N_α can be written in terms of S,

$$\lambda_\alpha^{(n)} = \frac{S_{\alpha n}}{S_{0n}}. \tag{B2}$$

S also has the remarkable property that the largest eigenvalue of N_α, which is the quantum dimension d_α, is given by $\lambda_\alpha^{(0)}$,

$$d_\alpha = \frac{S_{0\alpha}}{S_{00}}. \tag{B3}$$

Inserting Eq. (B1) into Eq. (52) yields

$$\text{GSD} = \sum_{n=0}^{N-1} \left(\sum_{\alpha=0}^{N-1} \lambda_\alpha^{(n)}\lambda_{\bar\alpha}^{(n)} \right)^{g-1}$$

$$= \sum_{n=0}^{N-1} (S_{0n})^{-2(g-1)}\left(\sum_{\alpha=0}^{N-1} S_{\alpha n}S_{\bar\alpha n} \right)^{g-1}. \tag{B4}$$

Using the fact that $S_{\alpha\beta}S_{\beta\gamma}=C_{\alpha\gamma}$ and $C_{\alpha\beta}C_{\beta\gamma}=\delta_{\alpha\gamma}$, we see that $\Sigma_\alpha S_{\alpha n}S_{\bar\alpha n}=\Sigma_{\alpha\beta}S_{n\alpha}C_{\alpha\beta}S_{\beta n}=1$, so that Eq. (52) can be rewritten as

$$\text{GSD} = \sum_{\gamma=0}^{N-1} S_{0\gamma}^{-2(g-1)} = \left(\sum_{\gamma=0}^{N-1} d_\gamma^2 \right)^{g-1} \sum_{\gamma=0}^{N-1} d_\gamma^{-2(g-1)}, \tag{B5}$$

where in the last equality we use the fact that $\Sigma_\alpha d_\alpha^2=S_{00}^{-2}$, which follows from Eq. (B3) and $\Sigma_\alpha S_{0\alpha}S_{\alpha 0}=1$.

From the magnetic algebra structure of the quasiparticles that was described in Sec. IIIB, we know that quasiparticles in the same representation of the magnetic algebra differ from each other by Abelian quasiparticles and thus they all have the same quantum dimension. Since there are $c_i q$ quasiparticles in the ith representation, we can see that Eq. (B5) can be rewritten as

$$\text{GSD} = q^g \left(\sum_i c_i d_i^2 \right)^{g-1} \left(\sum_i c_i d_i^{-2(g-1)} \right), \tag{B6}$$

where the sum over i is a sum over the different representations of the magnetic algebra and, as defined in Sec. IIIB, $c_i q$ is the dimension of the ith representation. Recall that $\nu = p/q$ with p and q coprime. d_i is the quantum dimension of the quasiparticles in the ith representation.

To proceed further, let us pause to consider the structure of the simple-current CFT. The simple-current CFT contains the "disorder" fields σ_i, which are primary with respect to the algebra generated by the simple current $\psi(z)$. There are also fields of the form $\psi^a \sigma_i$, which are primary with respect to the Virasoro algebra. Since $\psi^n = 1$, there are at most n different fields of the form $\psi^a \sigma_i$. However, these fields are not necessarily all distinct. It may be the case that σ_i and $\psi^a \sigma_i$ refer to the same field for certain values of a. This occurs when these two fields have the same pattern of zeros sequences. That is, when

$$l_{i;b}^{\text{sc}} = l_{i;a+b}^{\text{sc}} \tag{B7}$$

for $b = 0, \dots, n-1$. Let us suppose that this happens when a is a multiple of some integer c_i^{sc}. Then, $\psi^{c_i^{\text{sc}}} \sigma_i$ and σ_i label the same fields and so there are only c_i^{sc} distinct fields of the form $\psi^a \sigma_i$. Note that c_i^{sc} must divide n.

Now recall that the action of $\hat{T}_1$ on some quasiparticle $V_{i,\alpha} = \sigma_i e^{iQ_{(i,\alpha)}\sqrt{1/\nu}\varphi}$ is to take it to a new quasiparticle that differs from $V_{i,\alpha}$ by a U(1) factor,

$$\hat{T}_1 : \sigma_i e^{iQ_{(i,\alpha)}\sqrt{1/\nu}\varphi} \to \sigma_i e^{i(Q_{(i,\alpha)}+\nu)\sqrt{1/\nu}\varphi}. \tag{B8}$$

So if we apply $\hat{T}_1^{c_i q}$ to $V_{i,\alpha}$, we get

$$\hat{T}_1^{c_i q} : V_{i,\alpha} \to \sigma_i e^{i(Q_{(i,\alpha)}+c_i q \nu)\sqrt{1/\nu}\varphi}$$
$$= \sigma_i e^{i(Q_{(i,\alpha)}+c_i p)\sqrt{1/\nu}\varphi} \sim \psi^{n-c_i p} \sigma_i e^{iQ_{(i,\alpha)}\sqrt{1/\nu}\varphi}, \tag{B9}$$

where in the last step we have used the fact that two quasiparticles are equivalent if they differ by electron operators. Since $c_i q$ is the dimension of the ith representation, quasiparticles in the ith representation are invariant under the action of $\hat{T}_1^{c_i q}$. This means that $\sigma_i e^{iQ_{(i,\alpha)}\sqrt{1/\nu}\varphi} \sim \psi^{n-c_i p} \sigma_i e^{iQ_{(i,\alpha)}\sqrt{1/\nu}\varphi}$. This happens only when σ_i and $\psi^{n-c_i p} \sigma_i$ refer to the same fields. Since $\sigma_i \sim \psi^{c_i^{\text{sc}}} \sigma_i \sim \psi^{n-c_i^{\text{sc}}} \sigma_i$ all refer to the same field, we find that

$$c_i p = c_i^{\text{sc}}. \tag{B10}$$

Inserting this in Eq. (B6) yields Eq. (54).

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